JOURNAL OF ALGEBRA 148, 68–85 (1992)

Explicit Generalized Pieri Maps

MIHALIS MALIAKAS

Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701

AND

PETER J. OLVER*

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Communicated by David Buchsbaum

Received July 5, 1990

The purpose of this paper is to describe explicitly in a characteristic-free setting certain maps between skew Weyl modules for the general linear group. As an application, we indicate briefly how our generalized Pieri maps are utilized to obtain new results concerning nonzero homomorphisms between certain skew Weyl modules over fields of positive characteristics. In particular, we obtain a new proof of the special case of the Carter–Payne theorem considered in this paper. © 1992 Academic Press, Inc.

INTRODUCTION

Let V be a finite dimensional vector space over the field of rational numbers and let λ be a partition. By $K_\lambda V$ we denote the Weyl module for the general linear group $GL(V)$ corresponding to λ. The representation $K_\lambda V$ is an irreducible polynomial representation of $GL(V)$. Consider the tensor product $V \otimes K_\lambda V$ (over the rationals). Pieri's formula describes the decomposition of $V \otimes K_\lambda V$ into irreducible $GL(V)$ modules. We have

$$V \otimes K_\lambda V = \sum_{\lambda'} K_{\lambda'} V,$$

where λ' runs over all partitions obtained by adding one box to the

* Research supported in part by the National Science Foundation under Grant DMS 8901600.

0021-8693/92 $5.00

diagram of λ. Explicit maps of the type $\varphi_\lambda: V \otimes K_\lambda V \to K_{\lambda'} V$ were first constructed by the second-named author in [O_1, Sect. 5]. These maps were important in the study of "differential hyperforms" (see also [O_2, Sect. 6]). Another application of these maps appears in [D], where the invariant ideals of the symmetric algebra $S(V \oplus \Lambda^2 V)$ are studied. The purpose of this paper is to show that the Pieri maps φ_λ above can be extended in a characteristic-free way to obtain explicit maps between certain skew Weyl modules (See Theorem 1 and its corollary of Section 2). The usefulness of our construction is demonstrated in the last section of this paper, where new results concerning nonzero homomorphisms between certain skew Weyl modules over fields of positive characteristic are obtained (see Theorem 10 of Section 4). For simplicity our discussion there is restricted to a special case. As a by-product of our considerations we obtain a new proof of the corresponding special case of the Carter–Payne theorem [CP]. The general case will appear elsewhere [M].

1. Notation and Preliminaries

Let R be a commutative ring with identity and let F be a free R module of finite rank. The exterior algebra and the divided power algebra of F will be denoted by ΛF and DF, respectively. If $\alpha = (a_1, ..., a_s)$ is any sequence of nonnegative integers we denote the tensor product (over R) $\Lambda^{\alpha_1} F \otimes \Lambda^{\alpha_2} F \otimes \cdots \otimes \Lambda^{\alpha_s} F$ of exterior powers of F by $\Lambda(a)$. Likewise we put $D(\alpha) = D_{a_1} F \otimes \cdots \otimes D_{a_s} F$. If λ/μ is a skew partition, i.e., λ and μ are partitions $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0)$, $\mu = (\mu_1 \geqslant \cdots \geqslant \mu_k \geqslant 0)$ such that $\lambda_i \geqslant \mu_i$ for $i = 1, ..., k$, we denote by $\tilde{\lambda}/\tilde{\mu}$ the transpose skew partition of λ/μ. In [ABW] the Weyl module, $K_{\lambda/\mu} F$, for the general linear group $GL(F)$ was defined as the image of a particular natural map

$$d'_{\lambda/\mu}: D(\lambda/\mu) \to \Lambda(\tilde{\lambda}/\tilde{\mu}).$$

We will need two results from [ABW] concerning the modules $K_{\lambda/\mu} F$: (a) the standard basis theorem for $K_{\lambda/\mu} F$ and (b) the description of $K_{\lambda/\mu} F$ in terms of generators and relations. (See [ABW, Theorem II.3.16].) We state these facts.

(A) *Let S be an ordered basis for F. The module $K_{\lambda/\mu}$ is free over R with basis given by*

$$\{ d'_{\lambda/\mu}(X_T) \mid T \text{ is costandard tableau in } S \text{ of shape } \lambda/\mu \},$$

where X_T is the element of $D(\lambda/\mu)$ corresponding to T.

Before we state the second fact we establish some notation. If $(a_1, ..., a_k)$ is any sequence of nonnegative integers we define for $i<j$ and $t \geqslant 0$ a natural map

$$\sigma_{ij}^{(t)}: D(a_1, ..., a_k) \to D(a_1, ..., a_i - t, ..., a_j + t, ..., a_k)$$

as the composition of t-fold diagonalization on the ith factor

$$D(a_1, ..., a_k) \to D(a_1, ..., a_i - t, t, ..., a_k)$$

and multiplication of $D_t F$ with the jth factor of $D(a_1, ..., a_k)$,

$$D(a_1, ..., a_i - t, t, ..., a_j, ..., a_k) \to D(a_1, ..., a_i - t, ..., a_j + t, ..., a_k).$$

Similarly, if $i > j$ and $t \geqslant 0$, we define a natural map

$$\sigma_{ij}^{(t)}: D(a_1, ..., a_k) \to D(a_1, ..., a_j + t, ..., a_i - t, ..., a_k)$$

as the composition of the indicated diagonalization and multiplication. For $t=1$ we write $\sigma_{ij}^{(1)} = \sigma_{ij}$. Finally, if $i=j$ we put $\sigma_{ii} =$ identity map. Now if $\lambda/\mu = (\lambda_1 - \mu_1, ..., \lambda_k - \lambda_k)$ is a skew partition, define a map

$$\square_{\lambda/\mu}: \sum_{i,l} D(\lambda_1 - \mu_1, ..., \lambda_i - \mu_i + \lambda_{i+1} - \mu_{i+1} - l, l, ..., \lambda_k - \mu_k) \to D(\lambda/\mu),$$

where i ranges from 1 to $k-1$ and l ranges from 0 to $\lambda_{i+1} - \mu_i - 1$, by putting

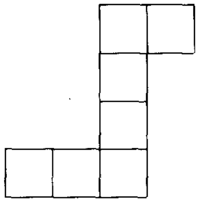
$$\square_{\lambda/\mu} = \sum_{i,l} \sigma_{ii+1}^{(\lambda_{i+1} - \mu_{i+1} - l)}.$$

(B) *There is an natural isomorphism* Coker $\square_{\lambda/\mu} = K_{\lambda/\mu} F$.

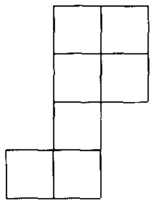
2. Generalized Pieri Maps

Let α and $\alpha(d, r)$ be two skew partitions related as follows: the diagram of $\alpha(d, r)$ is obtained from the diagram of α by subtracting d boxes from the left of the bottom row of α and adding these to the right of the rth row of α. Thus if $\alpha = (\lambda_1, ..., \lambda_k)/(\mu_1, ..., \mu_k)$, then $\alpha(d, r) = (\lambda_1, ..., \lambda_r + d, ..., \lambda_k)/(\mu_1, ..., \mu_k + d)$. Note that since α and $\alpha(d, r)$ are skew partitions we have $\mu_{k-1} \geqslant \mu_k + d$ and $\lambda_{r-1} \geqslant \lambda_r + d$. We always assume $r < k$.

EXAMPLE. If

$\alpha =$  ,

$d=1$, and $r=2$, then

$\alpha(1,2)=$  .

For $r=2$ we cannot take $d>1$ in this example.

Let k denote the number of rows of α and assume $d=1$. Assuming $\alpha(1,r)$ is a skew partition we want to define a natural map

$$\varphi_{\alpha,r}\colon D(\alpha)\to D(\alpha(1,r)),$$

which will induce a map of the corresponding Weyl modules. This will include the Pieri maps as a special case; see Remark 1 below. If $J=(r<j_1\cdots<j_q<k)$ is an ordered $(q+2)$-tuple of positive integers define the map

$$\sigma_J\colon D(a_1,...,a_k)\to D(a_1,...,a_r+1,...,a_k-1)$$

as the composition

$$\sigma_J=\sigma_{kj_q}\sigma_{j_qj_{q-1}}\cdots\sigma_{j_1r},$$

where the maps σ_{ij} were defined in Section 1. Also for J as above we define an integer c_J (associated to α) as a product of hook lengths given by the formula

$$c_J=\prod_i(\lambda_r-\lambda_i+i-r+1),$$

where i runs over all $r+1, r+2, ..., k$ such that $i\notin J$ and $\lambda_i\neq\mu_i$ (Perhaps a remark on the condition $\lambda_i\neq\mu_i$ is needed here. Any skew Weyl module

$K_{\lambda/\mu}F$ can be represented by λ/μ with $\lambda_i - \mu_i > 0$, in which case our condition is not needed.) Now put

$$\varphi_{\alpha,r} = \sum_J c_J \sigma_J,$$

where J runs over all sequences of the form $(r < j_1 < \cdots < j_q < k)$.

We can now state our main result, where we write φ_α for $\varphi_{\alpha,r}$.

THEOREM 1. *Let F be a free module over the commutative ring R and let α be a skew partition with k rows, where $k \leqslant \operatorname{rank} F$. The map $\varphi_\alpha \colon D(\alpha) \to D(\alpha(1,r))$ induces a map*

$$\tilde{\varphi}_\alpha \colon K_\alpha F \to K_{\alpha(1,r)} F.$$

If R is the ring of integers, $\tilde{\varphi}_\alpha$ is nonzero, and hence if R is a field of characteristic $p > 0$, the map $(1/\delta)\,\tilde{\varphi}_\alpha$ is nonzero, where δ is a suitable power of p.

Before we start the proof we have an example and two remarks.

EXAMPLE. Assume $r = 1$ in the definition of φ_α. For $k = 2$ we have

$$\varphi_\alpha = \sigma_{21},$$

while for $k = 3$

$$\varphi_\alpha = (\lambda_1 - \lambda_2 + 2)\,\sigma_{31} + \sigma_{32}\sigma_{21}$$

and for $k = 4$

$$\begin{aligned}\varphi_\alpha = {} & (\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_3 + 3)\,\sigma_{41} + (\lambda_1 - \lambda_2 + 2)\,\sigma_{43}\sigma_{31} \\ & + (\lambda_1 - \lambda_3 + 3)\,\sigma_{42}\sigma_{21} + \sigma_{43}\sigma_{32}\sigma_{21}.\end{aligned}$$

Remarks. 1. If $\alpha = (\lambda_1 + 1, \lambda_2 + 1, ..., \lambda_{k-1} + 1, 1)/(1^{k-1})$ we have $K_\alpha F = F \otimes K_\lambda F$ and $K_{\alpha(1,r)} F = K_{\lambda'} F$, where $\lambda' = (\lambda_1, ..., \lambda_r + 1, ..., \lambda_{k-1})$ and thus we obtain the Pieri maps

$$F \otimes K_\lambda F \to K_{\lambda'} F.$$

If, moreover, R is a field of characteristic zero our maps $F \otimes K_\lambda F \to K_{\lambda'} F$ are integer multiples of the maps constructed in [O_1, Sect. 5], the integer being

$$\prod_{j=r+1}^{k-1} (\lambda_r - \lambda_j + j - s + 1),$$

which serves to clear the denominators.

2. The assumption on the rank of F in the statement of the theorem is used only to ensure that the corresponding Weyl modules are nonzero, and we note that this is the weakest possible condition on rank F by (A) of Section 1.

We use the notation $[\sigma_{ij}^{(s)}, \sigma_{mn}^{(t)}] := \sigma_{ij}^{(s)}\sigma_{mn}^{(t)} - \sigma_{mn}^{(t)}\sigma_{ij}^{(s)}$ for the commutator of the maps $\sigma_{ij}^{(s)}$ and $\sigma_{mn}^{(t)}$. For the proof of the theorem we will need the following very easy lemma.

LEMMA 2. *Let* $(a_1, ..., a_k)$ *be a sequence of positive integers. For* $x = x_1 \otimes \cdots \otimes x_k \in D(a_1, ..., a_k)$, $i \neq j$, *and* $m \neq n$ *we have*

$$[\sigma_{ij}, \sigma_{mn}^{(t)}](x) = \begin{cases} (a_j - a_i - t + 1)\, \sigma_{mn}^{(t-1)}(x) & \text{if } i = n \text{ and } j = m \\ \sigma_{mj}\sigma_{mn}^{(t-1)}(x) & \text{if } i = n \text{ and } j \neq m \\ -\sigma_{in}\sigma_{mn}^{(t-1)}(x) & \text{if } i \neq n \text{ and } j = m \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) For the first case ($i = n$ and $j = m$) it is enough to check the desired equality for $i = n = 1$, $j = m = 2$. Now, if we denote multiplication in DF by juxtaposition and if the image of $x_2 \in D(a_2)$ under the diagonalization $D(a_2) \to D(t, a_2 - t)$ is denoted by $\sum_\alpha x_{2\alpha}(t) \otimes x_{2\alpha}(a_2 - t)'$, we have

$$\sigma_{21}^{(t)}(x_1 \otimes x_2) = \sum_\alpha x_1 x_{2\alpha}(t) \otimes x_{2\alpha}(a_2 - t)'.$$

Thus

$$\begin{aligned} \sigma_{12}\sigma_{21}^{(t)}(x) = {} & \sum_{\alpha,\beta} x_{1\beta}(a_1 - 1)\, x_{2\alpha}(t) \otimes x_{1\beta}(1)'\, x_{2\alpha}(a_2 - t)' \\ & + (a_2 - t + 1) \sum_\alpha x_1 x_{2\alpha}(t-1) \otimes x_{2\alpha}(a_2 - t + 1)', \end{aligned}$$

where the coefficient $a_2 - t + 1$ comes from multiplication in DF. Likewise we have

$$\begin{aligned} \sigma_{21}^{(t)}\sigma_{12}(x) = {} & \alpha_1 \sum_\alpha x_1 x_{2\alpha}(t-1) \otimes x_{2\alpha}(a_2 - t + 1)' \\ & + \sum_{\alpha,\beta} x_{1\beta}(a_1 - 1)\, x_{2\alpha}(t) \otimes x_{1\beta}(1)'\, x_{2\alpha}(a_2 - t)'. \end{aligned}$$

Hence $[\sigma_{12}, \sigma_{21}^{(t)}](x) = (a_2 - a_1 - t + 1)\, \sigma_{21}^{(t-1)}(x)$, as desired.

(2), (3), (4). The proofs are similar and thus omitted.

Proof of Theorem 1.

We first prove the theorem for $r=1$. The general case will follow easily from this as we will see later. Recall from (B) of Section 1 that the relations of $K_\alpha F$ are indexed by $i=1, 2, ..., k-1$. Consider the relation

$$D(\lambda_1-\mu_1, ..., \lambda_i-\mu_i+\lambda_{i+1}-\mu_{i+1}-l, l, ..., \lambda_k-\mu_k)$$

for $0 \leqslant l < \lambda_{i+1}-\mu_i$. We distinguish two cases.

Case I. $i<k-1$. Define a map ψ_α on the relations

$$\begin{aligned}\psi_\alpha\colon D(\lambda_1-\mu_1, ..., \lambda_i-\mu_i+\lambda_{i+1}-\mu_{i+1}-l, l, ..., \lambda_k-\mu_k)&\\ \to D(\lambda_1+1-\mu_1, \lambda_2-\mu_2, ..., \lambda_i-\mu_i&\\ +\lambda_{i+1}-\mu_{i+1}-l-1, l+1, ..., \lambda_k-\mu_k-1)&\end{aligned}$$

by

$$\psi_\alpha=(\lambda_{i+1}-\mu_i-l-1)\sum_{J'} c_{J'}\sigma_{J'_2}\sigma_{J'_1},$$

where the sum ranges over all $J'=(1<j_1<\cdots<j_q<k)$ that contain both i and $i+1$, and J'_1, J'_2 are defined as follows: $J'_1=(1<j_1<\cdots<i)$ and $J'_2=(i+1<\cdots<j_q<k)$. (Note $J'_1\cup J'_2=J'$ and $J'_1\cap J'_2=\phi$.) The coefficients $c_{J'}$ were defined earlier in this section. We remark that for $i=1$ the target of ψ_α is $D(\lambda_1-\mu_1+\lambda_2-l,\ l+1,\ \lambda_3-\mu_3, ..., \lambda_k-\lambda_k)$. In this case $\sigma_{J'_1}=\sigma_{11}$, which is the identity map by definition.

Note that the range of ψ_α is a relation for $K_{\alpha(1,1)}F$ except when $l=\lambda_{i+1}-\mu_i-1$. But in this case $\psi_\alpha=0$. Thus we consider the diagram

$$\begin{tikzcd}
D(\lambda_1-\mu_1, ..., \lambda_i-\mu_i+\lambda_{i+1}-\mu_{i+1}-l, l, ..., \lambda_k-\mu_k) \arrow[r, "a"] \arrow[d, "\varphi_\alpha+\psi_\alpha"] & D(\alpha) \arrow[d, "\varphi_\alpha"] \\
D(\lambda_1+1-\mu_1, ..., \lambda_i-\mu_i+\lambda_{i+1}-\mu_{i+1}-l, l, ..., \lambda_k-\mu_k-1) \arrow[r, "a"] & D(\alpha(1,1)) \\
+ & \\
D(\lambda_1+1-\mu_1, ..., \lambda_i-\mu_i+\lambda_{i+1}-\mu_{i+1}-l-1, l+1, ..., \lambda_k-\mu_k-1) \arrow[uur, "b"'] &
\end{tikzcd}$$

where $a=\sigma_{ii+1}^{(\lambda_{i+1}-\mu_{i+1}-l)}$ and $b=\sigma_{ii+1}^{(\lambda_{i+1}-\mu_{i+1}-l-1)}$.

Note that the maps a and b above are the appropriate components of the maps $\square_\alpha$ and $\square_{\alpha(1,1)}$ that were considered in (B) of Section 1. We claim that our diagram commutes, i.e., $[\varphi_\alpha, a]=b\psi_\alpha$. Indeed, let J be a sequence $(1<\cdots<k)$. Four cases need to be considered.

1. $i \in J$ and $i+1 \in J$. We write

$$J = J' = (1 < j_1 < \cdots < j_s < i < i+1 < j_{s+1} < \cdots < j_q < k).$$

2. $i \notin J$ and $i+1 \in J$. We write

$$J = J'' = (1 < j_1 < \cdots < j_s < i+1 < j_{s+1} < \cdots < j_q < k)$$

3. $i \in J$ and $i+1 \notin J$. We write

$$J = J''' = (1 < j_1 < \cdots < j_s < i < j_{s+1} < \cdots < j_q < k).$$

4. $i \notin J$ and $i+1 \notin J$. We write

$$J = (1 < j_1 < \cdots < j_s < j_{s+1} < \cdots < j_q < k).$$

Now in each case we compute and apply the lemma.

1. $$[\sigma_{J'}, a] = \sigma_{kj_q} \cdots \sigma_{j_{s+1}i+1} [\sigma_{i+1i}, \sigma_{ii+1}^{(\lambda_{i+1} - \mu_{i+1} - l)}] \, \sigma_{ij_s} \cdots \sigma_{j_1 1}$$
$$= (\lambda_i - \mu_i - l) \, \sigma_{kj_q} \cdots \sigma_{j_{s+1}i+1} \sigma_{ii+1}^{(\lambda_{i+1} - \mu_{i+1} - l - 1)} \sigma_{ij_s} \cdots \sigma_{j_1 1}.$$
2. $$[\sigma_{J''}, a] = \sigma_{kj_q} \cdots \sigma_{j_{s+1}i+1} [\sigma_{i+ij_s}, \sigma_{ii+1}^{(\lambda_{i+1} - \mu_{i+1} - l - 1)}] \, \sigma_{j_s j_{s-1}} \cdots \sigma_{j_1 1}$$
$$= \sigma_{kj_q} \cdots \sigma_{j_{s+1}i+1} \sigma_{ij_s} \sigma_{ii+1}^{(\lambda_{i+1} - \mu_{i+1} - l)} \sigma_{j_s j_{s-1}} \cdots \sigma_{j_1 1}.$$
3. $$[\sigma_{J'''}, a] = \sigma_{kj_q} \cdots \sigma_{j_{s+2} j_{s+1}} [\sigma_{j_{s+1}i}, \sigma_{ii+1}^{(\lambda_{i+1} - \mu_{i+1} - l)}] \, \sigma_{ij_s} \cdots \sigma_{j_1 1}$$
$$= -\sigma_{kj_q} \cdots \sigma_{j_{s+2} j_{s+1}} \sigma_{j_{s+1}i+1} \sigma_{ii+1}^{(\lambda_{i+1} - \mu_{i+1} - l - 1)} \sigma_{ij_s} \cdots \sigma_{j_1 1}.$$
4. $$[\sigma_J, a] = 0.$$

Thus

$$[c_{J'}\sigma_{J'} + c_{J''}\sigma_{J''} + c_{J'''}\sigma_{J'''} + c_J \sigma_J, a]$$
$$= \{(\lambda_i - \mu_i - l) \, c_{J'} + c_{J''} - c_{J'''}\}$$
$$\times \sigma_{kj_q} \cdots \sigma_{j_{s+1}i+1} \sigma_{ii+1}^{(\lambda_{i+1} - \lambda_{i+1} - l - 1)} \sigma_{ij_s} \cdots \sigma_{j_1 1}. \tag{1}$$

From the definition of the integers c_J we have

$$c_{J''} = (\lambda_1 - \lambda_i + i - 1) \, c_{J'}$$

and

$$c_{J'''} = (\lambda_1 - \lambda_{i+1} + i) \, c_{J'}.$$

By substituting in (1) and summing over all sequences J of the type considered in case 4 and recalling the definition of the map ψ_α we see that $[\varphi_\alpha, a] = b\psi_\alpha$ and hence our diagram commutes. This finishes case I.

Case II. $i=k-1$. We need to define a different map on the relations in this case but the idea is similar. Consider the relation $D(\lambda_1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l, l)$ for $K_\alpha F$, where $0 \leqslant l < \lambda_k-\mu_{k-1}$. Note that both $D(\lambda_1+1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l, l-1)$ and $D(\lambda_1+1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l-1, l)$ are relations for $K_{\alpha(1,1)}F$. (If in the first of these $l=0$, then this relation is zero.)

Define a map

$$\begin{aligned}\psi'_\alpha\colon & D(\lambda_1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l, l)\\ &\to D(\lambda_1+1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l-1, l)\end{aligned}$$

by setting

$$\psi'_\alpha=(\lambda_1-\mu_{k-1}+k-l-1)\sum_{J'} c_{J'}\sigma_{J'_1},$$

where J' ranges over all sequences of the form $(1<j_1<\cdots<k-1<k)$ and $J'_1=J'-\{k\}$. We claim that the diagram

$$\begin{tikzcd}
D(\lambda_1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l, l) \arrow[r, "a'"] \arrow[d, "\varphi_\alpha+\psi'_\alpha"] & D(\alpha) \arrow[d, "\varphi_\alpha"] \\
D(\lambda_1+1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l, l-1) \arrow[r, "a'"] & D(\alpha(1,1)) \\
+ & \\
D(\lambda_1+1-\mu_1, ..., \lambda_{k-1}-\mu_{k-1}+\lambda_k-\mu_k-l-1, l) \arrow[uur, "b'"'] &
\end{tikzcd} \qquad (*)$$

commutes. Here a' and b' are the appropriate components of the maps $\square_\alpha$ and $\square_{\alpha(1,1)}$ considered in (B) of Section 1, i.e., $a'=\sigma_{k-1k}^{(\lambda_k-\mu_k-l)}$ and $b'=\sigma_{k-1k}^{(\lambda_k-\mu_k-l-1)}$. To prove the claim let $J=(1<\cdots<k)$. Two cases need to be considered.

1. $k-1\in J$. We write $J=J'=(1<j_1<\cdots<j_q<k-1<k)$.
2. $k-1\notin J$. We write $J=(1<j_1<\cdots<j_q<k)$.

Again using the lemma we have

1. $[\sigma_{J'}, a']=(\lambda_{k-1}-\mu_{k-1}-l)\,\sigma_{k-1k}^{(\lambda_k-\mu_k-l-1)}\sigma_{k-1j_q}\cdots\sigma_{j_1 1}$.
2. $[\sigma_{J'}, a']=\sigma_{k-1k}^{(\lambda_k-\mu_k-l-1)}\sigma_{k-1j_q}\cdots\sigma_{j_1 1}$.

Thus

$$\begin{aligned}&[c_{J'}\sigma_{J'}+c_J\sigma_J, a']\\ &\quad=\{(\lambda_{k-1}-\mu_{k-1}-l)\,c_{J'}+c_J\}\,\sigma_{k-1k}^{(\lambda_k-\mu_k-l-1)}\sigma_{k-1j_q}\cdots\sigma_{j_1 1}.\end{aligned} \qquad (2)$$

From the definition of the coefficients c_J,

$$c_J = (\lambda_1 - \lambda_{k-1} + k - 1)\, c_{J'}.$$

Substituting in (2) and summing over all J' of the type considered in case 1 we have $[\varphi_{\alpha'}, a'] = b'\psi'_\alpha$ and this concludes the proof of case II.

So far we have proved that the map $\varphi_\alpha\colon D(\alpha) \to D(\alpha(1,1))$ induces a map of skew Weyl modules $K_\alpha F \to K_{\alpha(1,1)} F$. Now consider the map

$$1 \otimes \cdots \otimes 1 \otimes \varphi_\beta\colon D(\alpha) \to D(\alpha(1, r)),$$

which is the identity on the first $r-1$ factors of $D(\alpha)$ and φ_β, where β is the skew shape $(\lambda_r - \mu_r, ..., \lambda_k - \mu_k)$, on the other factors. It follows easily that this map induces a map

$$K_\alpha F \to K_{\alpha(1,r)} F.$$

Indeed, by the previous situation $(r = 1)$ we need only consider relations coming from the first $r-1$ rows of $D(\alpha)$ (i.e., $i = 1, 2, ..., r-1$). For $i = 1, 2, ..., r-2$ the map on the relations is given by

$$\begin{aligned} 1 \otimes \cdots \otimes 1 \otimes \varphi_\beta\colon & D(\lambda_1 - \mu_1, ..., \lambda_i - \mu_i + \lambda_{i+1} - \mu_{i+1} \\ & - l, l, ..., \lambda_r - \mu_r, ..., \lambda_k - \mu_k) \\ & \to D(\lambda_1 - \mu_{11}, ..., \lambda_i - \mu_i + \lambda_{i+1} - \mu_{i+1} \\ & - l, l, ..., \lambda_r + 1 - \mu_r, ..., \lambda_k - 1 - \mu_k). \end{aligned}$$

Finally for $i = r-1$ the map on the relations is given by

$$\begin{aligned} 1 \otimes \cdots \otimes 1 \otimes \varphi_\beta\colon & D(\lambda_1 - \mu_1, ..., \lambda_{r-1} - \mu_{r-1} \\ & + \lambda_r - \mu_r - l, l, ..., \lambda_k - \mu_k) \\ & \to D(\lambda_1 - \mu_1, ..., \lambda_{r-1} - \mu_{r-1} \\ & + \lambda_r - \mu_r - l, l+1, ..., \lambda_k - 1 - \mu_k). \end{aligned}$$

Thus we have proved that the map $\varphi_\alpha\colon D(\alpha) \to D(\alpha(1, r))$ induces a map, $\tilde{\varphi}_\alpha\colon K_\alpha F \to K_{\alpha(1,r)} F$. Now we will show that over the integers the map $\tilde{\varphi}_\alpha$ is nonzero. Consider the tableau T of shape α whose entries in the ith row are all equal to i. Let $I = (r, k)$ and let $X_T \in D(\alpha)$ be the element corresponding to T. Then

$$\varphi_\alpha(X_T) = c_I \sigma_I(X_T) + \sum_{J \neq I} c_J \sigma_J(X_T).$$

Now note: (a) every $\sigma_J(X_T)$ is row costandard, i.e., weakly increasing along the rows; (b) for every $J \neq I$, $\sigma_J(X_T)$ is lexicographically strictly smaller than $\sigma_I(X_T)$; and (c) $\sigma_I(X_T)$ is costandard, i.e., a basis element of $K_{\alpha(1,r)}F$ (see (A) of Section 1). These observations together with the fact that the straightening law decreases the lexicographic order of row costandard tableaux (see [ABW, Lemma II.3.15]) implies that $\tilde{\varphi}_\alpha \neq 0$ when the ground ring, R, is the ring of integers. This concludes the proof of the theorem.

COROLLARY 3. *The composition of maps*

$$D(\alpha) \xrightarrow{\varphi_\alpha} D(\alpha(1,r)) \xrightarrow{\varphi_{\alpha(1,r)}} D(\alpha(2,r)) \longrightarrow \cdots \longrightarrow D(\alpha(d,r))$$

induces a map $K_\alpha F \to K_{\alpha(d,r)}F$ which over the integers is nonzero.

Proof. To show that the induced map $K_\alpha F \to K_{\alpha(d,r)}F$ is nonzero over the integers one argues as in the proof of the theorem.

3. Commutativity of Maps

Let α be a skew partition with at least two boxes in the last row. Assume

$$\beta = \alpha(1,r), \qquad \beta' = \alpha(1,s), \qquad \text{and} \qquad \gamma = \beta(1,s) = \beta'(1,r)$$

are also skew partitions. The preceding maps give two distinct ways of mapping $K_\alpha F$ to $K_\gamma F$, according to the diagram

$$\begin{array}{ccc} K_\alpha F & \xrightarrow{\tilde{\varphi}_{\alpha,r}} & K_\beta F \\ {\scriptstyle \tilde{\varphi}_{\alpha,s}}\downarrow & & \downarrow{\scriptstyle \tilde{\varphi}_{\beta,s}} \\ K_{\beta'} F & \xrightarrow{\tilde{\varphi}_{\beta',r}} & K_\gamma F \end{array} \qquad (*)$$

and we are interested in whether the diagram commutes.

THEOREM 4. *Suppose α is a skew partition with at least two boxes in the last row. Assume $\beta = \alpha(1,r)$, $\beta' = (1,s)$ and $\gamma = \beta(1,s) = \beta'(1,r)$ are also skew partitions. Then the diagram $(*)$ commutes, i.e.,*

$$\tilde{\varphi}_{\beta,s}\tilde{\varphi}_{\alpha,r} = \tilde{\varphi}_{\beta',r}\tilde{\varphi}_{\alpha,s}. \qquad (**)$$

This commutativity is intimately related to the general definition of a "hypercomplex" introduced in $[O_1, O_2]$, significantly generalizing the Koszul and deRham complexes.

To prove Theorem 4, we begin by slightly generalizing the maps introduced in Section 2. Let $I=(i_1<i_2<\cdots<i_p)$ be an ordered p-tuple of positive integers. For $J=(j_1<j_2<\cdots<j_q)$ an ordered q-tuple of positive integers, we say $J\prec I$ if all the integers in J are contained in I, *and* the first and last entries agree: $i_1=j_1$, $i_p=j_q$. In this case, given $c=(c_1,...,c_N)\in\mathbf{Z}^N$, where $N\geqslant i_p$, define $c_J=\prod c_i$, where the product is over all $i\in I$ with $i\notin J$. Define the map

$$\varphi_I(c)=\sum_{J\prec I} c_J\sigma_J,$$

generalizing the Pieri map of Section 2. Note that $\varphi_I(c)$ does not actually depend on c_{i_1} or c_{i_p}. The following two lemmas are easily proved using Lemma 2.

LEMMA 5. *Let $j<k<i_1<\cdots<i_p$ be positive integers. Then*

$$[\sigma_{kj},\varphi_I(c)]=0,$$
$$[\sigma_{kj},\varphi_{kI}(c)]=\varphi_{jI}(c).$$

(*By kI we mean the $(p+1)$-tuple $(k<i_1<\cdots<i_p)$.*)

LEMMA 6. *Let $j<k<i_1<\cdots<i_n$ be positive integers. Then*

$$\varphi_{jkI}(c)=c_k\varphi_{jI}(c)+\varphi_{kI}(c)\sigma_{kj}.$$

In fact, the identity in Lemma 6 in a special case of the more general identity

$$\varphi_{JkI}(c)=c_k\varphi_{JI}(c)+\varphi_{kI}(c)\,\varphi_{Jk}(c)$$

valid for any $j_1<\cdots<j_q<k<i_1<\cdots<i_p$. However, we use only the elementary case here.

The key identity required to prove the commutativity theorem is the following.

LEMMA 7. *Let $j\leqslant k<i_1<\cdots<i_p$ be positive integers, $c\in\mathbf{Z}^N$ an N-tuple of integers with $N\geqslant i_p$, and $x\in\mathbf{Z}$ any integer. Define $c+x=(c_1+x, c_2+x, ..., c_N+x)$. Then*

$$x[\varphi_{jI}(c),\varphi_{kI}(c+x)]=\varphi_{jI}(c+x)\,\varphi_{kI}(c)-\varphi_{jI}(c)\,\varphi_{kI}(c+x).$$

COROLLARY 8. *For $j<i_1<\cdots<i_p$, and any $c\in\mathbf{Z}^N$, $x\in\mathbf{Z}$,*

$$[\varphi_{jI}(c),\varphi_{jI}(c+x)]=0.$$

Proof. Let $\theta(x) = [\varphi_{jI}(c), \varphi_{jI}(c+x)]$. Setting $j=k$ in the lemma, we find

$$x\theta(x) = -\theta(x)$$

which, as x is arbitrary, implies $\theta(x) = 0$.

Proof of Lemma 7. The identity is trivial if $I = (i_1)$ is a singleton. We prove it in general by induction on p, the number of elements of I. Thus, assuming the identity holds for any p-tuple I (and hence the corollary), the goal is to prove it for a $(p+1)$-tuple $\tilde{I} = (l, I)$, where $j \leqslant k < l < i_1 < \cdots < i_p$. Expanding the left hand side using Lemmas 5 and 6, we find

$$\begin{aligned}
&x[\varphi_{jlI}(c), \varphi_{klI}(c+x)] \\
&\quad = x[c_l\varphi_{jI}(c) + \varphi_{lI}(c)\sigma_{lj}, (c_l + x)\,\varphi_{kI}(c+x) + \varphi_{lI}(c+x)\sigma_{lk}] \\
&\quad = xc_l(c_l+x)[\varphi_{jI}(c), \varphi_{kI}(c+x)] + x(c_l+x)[\varphi_{lI}(c), \varphi_{kI}(c+x)]\,\sigma_{lj} \\
&\qquad + xc_l[\varphi_{jI}(c), \varphi_{lI}(c+x)]\,\sigma_{lk} + x[\varphi_{lI}(c), \varphi_{lI}(c+x)]\,\sigma_{lj}\sigma_{lk} \\
&\qquad + x(\varphi_{lI}(c)\,\varphi_{jI}(c+x)\,\sigma_{lk} - \varphi_{lI}(c+x)\,\varphi_{kI}(c)\sigma_{lj}).
\end{aligned}$$

On the other hand, the right hand side expands to

$$\begin{aligned}
&\varphi_{jlI}(c+x)\,\varphi_{klI}(c) - \varphi_{jlI}(c)\,\varphi_{klI}(c+x) \\
&\quad = ((c_l+x)\,\varphi_{jI}(c+x) + \varphi_{lI}(c+x)\sigma_{lj})(c_l\varphi_{kI}(c) + \varphi_{lI}(c)\sigma_{lk}) \\
&\qquad - (c_l\varphi_{jI}(c) + \varphi_{lI}(c)\sigma_{lj})((c_l+x)\,\varphi_{kI}(c+x) + \varphi_{lI}(c+x)\sigma_{lk}) \\
&\quad = c_l(c_l+x)(\varphi_{jI}(c+x)\,\varphi_{kI}(c) - \varphi_{jI}(c)\,\varphi_{kI}(c+x)) \\
&\qquad + (c_l+x)(\varphi_{lI}(c+x)\,\varphi_{kI}(c) - \varphi_{lI}(c)\,\varphi_{kI}(c+x))\sigma_{lj} \\
&\qquad + c_l(\varphi_{jI}(c+x)\,\varphi_{lI}(c) - \varphi_{jI}(c)\,\varphi_{lI}(c+x))\sigma_{lk} \\
&\qquad + x(\varphi_{jI}(c+x)\,\varphi_{lI}(c)\,\sigma_{lk} - \varphi_{lI}(c+x)\,\varphi_{kI}(c)\sigma_{lj}) \\
&\qquad + (\varphi_{lI}(c+x)\,\varphi_{jI}(c)\,\sigma_{lk} - \varphi_{lI}(c)\,\varphi_{jI}(c+x)\sigma_{lk}).
\end{aligned}$$

Comparing these two expressions, we see that the first three sets of terms are equal owing to the induction hypothesis for the indices (j, k, I), (k, l, I), and (j, l, I) respectively. The fourth commutator on the left hand side vanishes owing to the corollary. Finally, after one cancellation, the remaining terms reproduce another copy of the identity for the indices (j, l, I). Thus the induction step is complete and the identity proved.

Let $e_i = (0, \ldots 0, 1, 0, \ldots)$ denote the ith basis vector of $\mathbf{Z}^N$, so by $c + e_i$ we mean the p-tuple of integers obtained by increasing the ith element of c by 1. The commutativity theorem is a special case of the following result.

PROPOSITION 9. *Let* $j_1 < \cdots < j_q < i_1 < \cdots < i_p$ *be positive integers. Then, for any* $c \in \mathbf{Z}^{p+q}$, $x \in \mathbf{Z}$,

$$\varphi_I(c)\,\varphi_{JI}(c+x+e_{i_1}) = \varphi_{JI}(c+x)\,\varphi_I(c)$$

provided $c_{i_1} = 0$.

Proof. We begin with the case $J=(j)$ is a singleton. For simplicity, we replace I by (l, I), so $j<l<i_1<\cdots<i_p$. We need to show

$$\begin{aligned}
0 &= \varphi_{lI}(c)\,\varphi_{jlI}(c+x+e_l) - \varphi_{jlI}(c+x)\,\varphi_{lI}(c)\\
&= \varphi_{lI}(c)((c_l+x+1)\,\varphi_{jI}(c+x) + \varphi_{lI}(c+x)\sigma_{lj})\\
&\quad - ((c_l+x)\,\varphi_{jI}(c+x) + \varphi_{lI}(c+x)\sigma_{lj})\,\varphi_{lI}(c)\\
&= (c_l+x)[\varphi_{lI}(c), \varphi_{jI}(c+x)]\\
&\quad + (\varphi_{lI}(c)\,\varphi_{jI}(c+x) - \varphi_{lI}(c+x)\,\varphi_{jI}(c)) + [\varphi_{lI}(c), \varphi_{lI}(c+x)]\sigma_{lj}.
\end{aligned}$$

Here, the second equality follows since neither $\varphi_{jI}(c)$ nor $\varphi_{lI}(c)$ depend explicitly on the lth entry of c. Thus, assuming $c_l=0$, the result follows using Lemma 7 and Corollary 8. The general identity follows by an easy induction on the number of entries in J using Lemma 6.

To see that Proposition 9 includes the Commutativity Theorem, we assume $r<s<k$, and set $I=(s, s+1, ..., k)$, $J=(r, r+1, ..., s-1)$. Define

$$c_i = \lambda_s - \lambda_i + i - s + 1, \qquad i \neq s,$$
$$c_s = 0,$$

and

$$x = \lambda_r - \lambda_s + s - r.$$

Then note that

$$c_i + x = \lambda_r - \lambda_i + i - r + 1, \qquad i \neq s.$$

We find, then, that

$$\varphi_{\alpha,r} = \varphi_{JI}(c+x+e_s), \qquad \varphi_{\alpha,s} = \varphi_I(c),$$
$$\varphi_{\beta,s} = \varphi_I(c), \qquad \varphi_{\beta',r} = \varphi_{JI}(c+x),$$

since β' has one extra box in the sth row. The theorem follows.

Remark. A different proof can be constructed in the special case

$$\alpha = (\lambda_1+2, ..., \lambda_{k-1}+2, 2)/(2^{k-1})$$

corresponding to the Pieri formula for $K_\alpha F = D_2 F \otimes K_\lambda F$. According to the Littlewood–Richardson rule, the irreducible Weyl module $K_\gamma F$ appears exactly once in the decomposition of $K_\alpha F$ into irreducible components, except in the case $r = s \pm 1$, $\lambda_r = \lambda_s$. For this particular case, there are no nonzero natural maps from $K_\alpha F$ to $K_\gamma F$, so the composite map must be zero, and the theorem is trivial. Otherwise, there is, up to constant multiple, only one natural map, so the two composite maps in the formula must be multiples of each other. To prove the multiple is unity, it suffices to compute the highest lexicographic terms obtained by applying the maps to the canonical tableau X_T, where T has shape α and all entries in its ith row are i. The computation is not difficult, and also shows that the composite map is nonzero over the integers provided rank $F \geqslant k$.

Remark. The results in Lemmas 5, 6, 7, Corollary 8, and Proposition 9 are all consequences of the basic defining properties of the maps σ_{ij} contained in Lemma 2. It would be of great interest to determine the structure of the algebra generated by these maps in view of these rather remarkable identities contained therein. We remark that our algebra coincides with the algebra of polarization operators (applied to suitably homogeneous polynomials), which appear in the Capelli identity of invariant theory; see [H, pp. 565–566].

4. Application

In this section we illustrate how the explicit form of our generalized Pieri maps is utilized to obtain certain nonzero homomorphisms between skew Weyl modules over fields of positive characteristic. Here we restrict ourselves to a special case in order to keep our proofs short. This restriction however does not hinder the main idea, which is quite simple. The general case will be treated in [M]. We prove the following result.

THEOREM 10. *Let λ/γ and μ/γ be two skew partitions related as follows:*

$$\mu_i = \lambda_i + 1, \qquad \mu_j = \lambda_j - 1,$$
$$\mu_h = \lambda_h \qquad \text{for} \quad h \neq i, j \text{ for some } i < j.$$

Furthermore assume $\lambda_h - \lambda_{h+1} \geqslant 1$ for all $h = 1, ..., k-1$. Over a field of characteristic $p > 0$ we have

$$\mathrm{Hom}_{GL(F)}(K_{\lambda/\gamma} F, K_{\mu/\gamma} F) \neq 0$$

if p divides $\lambda_i - \lambda_j + j - i + 1$.

Remarks. 1. The hypothesis $\lambda_h - \lambda_{h+1} \geqslant 1$ is in fact not essential but it is assumed here in order to simplify our proofs. Also, one can prove an analogous result under the hypothesis $\mu_i = \lambda_i + d$ and $\mu_j = \lambda_j - d$.

2. For $\gamma=(0)$ we obtain the Carter–Payne result [CP] (with a new proof) corresponding to our special case. The general case will appear in [M].

Proof of Theorem 10. First, we prove the theorem for $i=1$ and $j=k$. The general situation will follow easily from this. Let α be the skew shape $(\lambda_1, ..., \lambda_{k-1}, \lambda_k - 1)/(\gamma_1, ..., \gamma_{k-1}, \gamma_k - 1)$. Then the skew shape $\alpha(1,1)$, as we recall from Section 2, is given by $(\lambda_1+1, ..., \lambda_{k-1}, \lambda_k-1)/(\gamma_1, ..., \gamma_{k-1}, \gamma_k)$. The first claim is that there is an exact sequence

$$d'_\alpha(X) \to K_\alpha F \to K_{\lambda/\gamma} F \to 0, \tag{1}$$

where d'_α is the map considered in the definition of $K_\alpha F$ (see Section 1) and X is the image of the map $\sigma_{k-1k}\colon D(\lambda_1-\gamma_1, ..., \lambda_{k-1}-\gamma_{k-1}+1, \lambda_k-\gamma_k-1) \to D(\alpha)$ that was defined in Section 1. Indeed, by (B) of Section 1 every relation of $K_\alpha F$ is also a relation of $K_{\lambda/\gamma}F$. Thus the identity map

$$D(\alpha) = D(\lambda/\gamma)$$

induces a surjection of the corresponding Weyl modules. Now note that the only relation of $K_{\lambda/\gamma}F$ which is not a relation for $K_\alpha F$ is the one corresponding to the pair of rows $k-1$ and k for $l=\lambda_k-\gamma_k-1$, i.e., it is given by the map

$$\sigma_{k-1k}\colon D(\lambda_1-\gamma_1, ..., \lambda_{k-1}-\gamma_{k-1}+1, \lambda_k-\gamma_k-1) \to D(\lambda/\gamma).$$

Therefore the exact sequence (1) is established.

Now from Section 2 recall that we have Pieri maps

$$\tilde{\varphi}_\alpha\colon K_\alpha F \to K_{\alpha(1,1)}F.$$

The second claim is that, under the hypothesis on the characteristic p, the restriction of $\tilde{\varphi}_\alpha$ on $d'_\alpha(X)$ is zero. Indeed, consider diagram (*) in the proof of Theorem 1 of Section 2 and let $l=\lambda_k-\gamma_{k-1}-1$ in this diagram. Then we have the commutative diagram

$$\begin{tikzcd}
D(\lambda_1-\gamma_1, ..., \lambda_{k-1}-\gamma_{k-1}+1, \lambda_k-\gamma_k-1) \arrow[r, "\sigma_{k-1k}"] \arrow[d, "\varphi_\alpha+\psi'_\alpha"] & D(\alpha) \arrow[d, "\varphi_\alpha"] \\
D(\lambda_1+1-\gamma_1, ..., \lambda_{k-1}-\gamma_{k-1}+1, \lambda_k-\gamma_k-2) \arrow[r, "\sigma_{k-1k}"] & D(\alpha(1,1)) \\
+ & \\
D(\lambda_1+1-\gamma_1, ..., \lambda_{k-1}-\gamma_{k-1}, \lambda_k-\gamma_k-1) \arrow[uur, "\mathrm{Id}"'] &
\end{tikzcd}$$

where Id stands for the identity map. The commutativity of this diagram and the definition of ψ'_α (case II in the proof of Theorem 1 of Section 2) imply that, for $x \in D(\lambda_1-\gamma_1, ..., \lambda_{k-1}-\gamma_{k-1}+1, \lambda_k-\gamma_k-1)$, we have

$$\begin{aligned}&\tilde{\varphi}_\alpha\, d'_\alpha \sigma_{k-1k}(x)\\ &\quad= d'_{\alpha(1,1)}\left\{\sigma_{k-1k}\varphi_\alpha(x)+(\lambda_1-\lambda_k+k)\left(\sum_{J'} c_{J'}\sigma_{J'_1}\right)(x)\right\}.\end{aligned}$$

Now since the map σ_{k-1k} in the bottom row of our diagram gives a relation for $K_{\alpha(1,1)}F$ (see (B) of Section 1) we have from the above equation

$$\tilde{\varphi}_\alpha\, d'_\alpha \sigma_{k-1k}(x)=(\lambda_1-\lambda_k+k)\, d'_{\alpha(1,1)}\left(\sum_{J'} c_{J'}\sigma_{J'_1}\right)(x).$$

Hence, under the hypothesis of the theorem that p divides $\lambda_1-\lambda_k+k$, we have

$$\tilde{\varphi}_\alpha(d'_\alpha(X))=0,$$

and the second claim is proven.

From the exact sequence (1) we see that the Pieri map

$$\tilde{\varphi}_\alpha\colon K_\alpha F \to K_{\alpha(1,1)}F$$

lifts, for characteristic p dividing $\lambda_1-\lambda_k+k$, to a map

$$\bar{\varphi}_{\lambda/\gamma}\colon K_{\lambda/\gamma}F \to K_{\alpha(1,1)}F.$$

Note that $\alpha(1,1)=\mu/\gamma$. Hence we have our natural map

$$\bar{\varphi}_{\lambda/\gamma}\colon K_{\lambda/\gamma}F \to K_{\mu/\gamma}F.$$

We show now that this map is nonzero. Consider the tableau, T, of shape λ/γ whose ith row entries are all equal to i, for $i=1, ..., k$. Observe the following: (a) Since (by the hypothesis) we have $\lambda_h-\lambda_{h+1}\geqslant 1$, for all h, the image of $X_T \in D(\lambda/\gamma)$ under the map $\sigma_J\colon D(\lambda/\gamma)\to D(\mu/\gamma)$ is again a costandard tableau for all $J=(1<\cdots<k)$; and (b) the coefficient of σ_I for $I=(1,2,...,k-1,k)$ in the definition of $\varphi_{\lambda/\gamma}$ (see Section 2) is equal to 1, i.e., $c_I=1$. Hence we see that $\bar{\varphi}_{\lambda/\gamma}\, d'_{\lambda/\gamma}(X_T)$ is a linear combination of distinct basis elements of $K_{\mu/\gamma}F$ and not all the coefficients are divisible by p. Thus our map is nonzero and this concludes the proof of the theorem for $i=1$ and $j=k$.

Now if i and j are arbitrary in the statement of the theorem consider the skew shapes $\beta=(\lambda_i, \lambda_{i+1}, ..., \lambda_j)/(\gamma_i, ..., \gamma_j)$ and $\beta'=(\mu_i, ..., \mu_j)/(\gamma_i, ..., \gamma_j)$. From the case that we just proved there is a nonzero map

$$\bar{\varphi}_\beta\colon K_\beta F \to K_{\beta'}F,$$

induced by the map

$$\varphi_\beta : D(\beta) \to D(\beta')$$

if p divides $\lambda_i - \lambda_j + j - i + 1$. Now consider the map

$$1 \otimes \cdots \otimes 1 \otimes \varphi_\beta \otimes 1 \otimes \cdots \otimes 1 : \; D(\lambda/\gamma) \to D(\mu/\gamma),$$

which is the identity on all rows except rows $i, i+1, ..., j$, where it is equal to φ_β. Just as in the proof (after case II) of Theorem 1 of Section 2 one sees very easily that we have corresponding maps on the relations and hence we obtain our induced natural map

$$K_{\lambda/\gamma} F \to K_{\mu/\gamma} F,$$

which is nonzero by the case $i = 1$ and $j = k$. This concludes the proof of Theorem 10.

Remark. As we mentioned before, the condition $\lambda_h - \lambda_{h+1} \geqslant 1$ is not essential. It was included here to show quickly that $\bar{\varphi}_{\lambda/\gamma} \neq 0$. Otherwise this is not true in general and one needs to consider maps of the form $(1/\delta)\, \bar{\varphi}_{\lambda/\gamma}$, where δ is a suitable power of the characteristic p.

Acknowledgments

The first-named author expresses his hearty thanks to Professor David Buchsbaum for suggesting the study of homomorphisms between skew Weyl modules in positive characteristics. He also acknowledges the hospitality of the School of Mathematics of the University of Minnesota, where part of this work was carried out.

References

[ABW] K. Akin, D. Buchsbaum, and J. Weynman, Schur functors and Schur complexes, *Adv. in Math.* **44** (1982), 207–278.

[CP] R. W. Carter and J. T. Payne, On homomorphisms between Weyl modules and Specht modules, *Math. Proc. Cambridge Philos. Soc.* **87** (1980), 419–425.

[D] A. Daszkiewicz, On the invariant ideals of the symmetric algebra $S(V \oplus \Lambda^2 V)$, *J. Algebra* **125** (1989), 444–473.

[H] R. Howe, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), 539–570.

[M] M. Maliakas, Skew Weyl modules and a theorem of Carter and Payne, in preparation.

[O_1] P. J. Olver, "Differential Hyperforms, I," University of Minnesota Mathematics Report 82–101, 1983.

[O_2] P. J. Olver, Invariant theory and differential equations, *in* "Invariant Theory" (S. S. Koh, Ed.), pp. 62–80, Lecture Notes in Mathematics, Vol. 1278, Springer-Verlag, New York/Berlin, 1987.